Introduction to Time Series Analysis

Lecture 2
Multivariate Time Series Models

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Recap

In the previous lecture we have seen:

- the tools needed when dealing with time series
- the most important properties of a stochastic process
- the most important classes of stochastic processes for time series analysis

But, the whole analysis was univariate, therefore the dynamic of a variable was explained using only

- its own lagged values
- the values of present and lagged exogenous shocks

Planned Schedule

In this lecture we will try to answer to the following questions:

- Why we need multivariate time series?
- Which multivariate model?
- VAR(p) models
- Cointegration

Why going multivariate? Theory (1)

Think about the standard Keynesian consumption function (static) or to the permanent income hypothesis:

$$c_t = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} y_t\right)$$

It is easy to see that:

- the change of a variable, affects the values of other variables.

Why going multivariate? Theory (2)

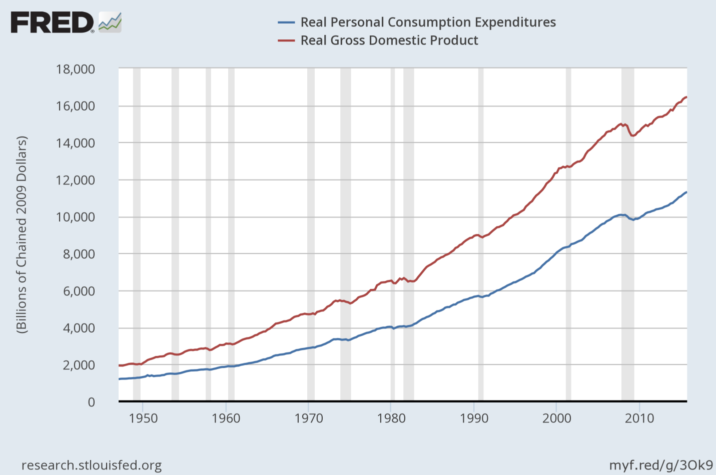
Think about a New-Keynesian (3-equations) Model:

$$\begin{aligned} x_t &= E_t\left[x_{t+1}\right] - \frac{1}{\sigma}\left(i_t - E_t\left[\pi_{t+1}\right]\right) + g_t && \text{IS} \\ \pi_t &= \beta E_t\left[\pi_{t+1}\right] + \lambda x_t + u_t && \text{NKPC} \\ i_t &= r_t^* + \varphi_\pi\left(\pi_t - \pi^*\right) + \varphi_x\left(x_t - x^*\right) && \text{TR} \end{aligned}$$

It is easy to see that:

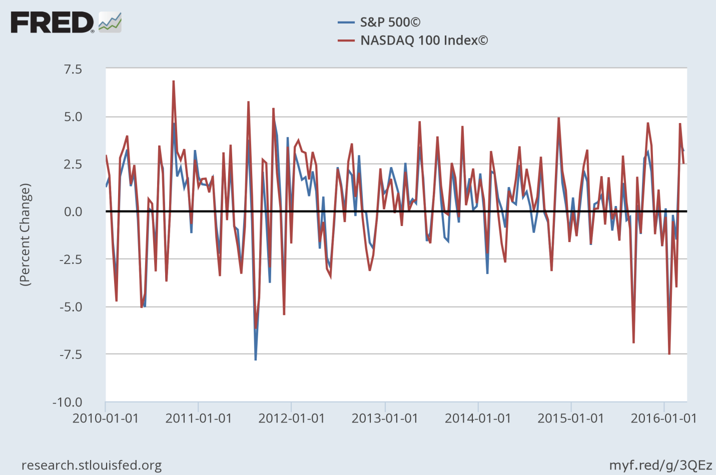
- the shock to a single variable, propagates also to the others.

Why going multivariate? Evidence (1)



Movements in consumption and income are clearly related.

Why going multivariate? Evidence (2)



Stock prices indexes are related as well.

Vector Autoregression VAR(p)

A strategy to take into account the fact that a variable's change might affect other variables dynamic is named **Vector Autoregression**

Let

$$\mathbf{x_t} = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{I,t} \end{bmatrix}$$

be a vector of time series, so that each single i:

$$x_{i,t} = (x_{i,1}, x_{i,2}, \dots, x_{i,T})', \quad \forall i = 1, \dots, I$$

is a stationary univariate time series.

Note: unless specified, here all the time series are considered to be stationary.

Vector Autoregression VAR(p)

We can therefore generalize the AR(p) process to the multivariate case and write:

$$\mathbf{x_t} = a_0 + A_1\mathbf{x_{t-1}} + A_2\mathbf{x_{t-2}} + \cdots + A_p\mathbf{x_{t-p}} + \varepsilon_\mathbf{t}$$

where:

- A_k: is a $m \times m$ matrix of parameters to be estimated;
- $\mathbf{x_{t-k}}$: are vectors of time series as specified in the previous slide;
- $\varepsilon_\mathbf{t}$: is a $m \times 1$ vector of random shocks

VAR(1) with 2 variables

To simplest case that allow us to understand what we are doing is a VAR(1) with 2 variables only.

$$\mathbf{x_t} = a_0 + A_1 \mathbf{x_{t-1}} + \varepsilon_{\mathbf{t}}$$

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} a_1^0 \\ a_2^0 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

and equation by equation we have

$$\begin{cases} x_{1,t} = a_1^0 + a_{11}^1 x_{1,t-1} + a_{12}^1 x_{2,t-1} + \varepsilon_{1,t} \\ x_{2,t} = a_2^0 + a_{21}^1 x_{1,t-1} + a_{22}^1 x_{2,t-1} + \varepsilon_{2,t} \end{cases}$$

Then, if both the variables are stationary, the model is efficiently estimated equation by equation using OLS or ML.

Stability of the VAR(1)

Stability: the property of a dynamical system to converge to its equilibrium value.

As with the AR(1) we can rewrite by means of iteration to get:

$$
\begin{aligned}
\mathbf{x_1} &= a_0 + A_1 x_0 + \varepsilon_1 \\
\mathbf{x_2} &= a_0 + A_1(a_0 + A_1 x_0 + \varepsilon_1) + \varepsilon_2 \\
&= \vdots \\
\mathbf{x_t} &= a_0(I + A_1 + A_1^2 + \cdots + A_1^{t-1}) + A_1^t x_0 + (\varepsilon_1 A_1^{t-1} + \cdots + \varepsilon_{t-1} A_1 + \varepsilon_t)
\end{aligned}
$$

In the AR(1) case, for the process to be stable we required $\lim_{t\to\infty} \lambda_1^t = 0$.

In the VAR(1) case, we require that the matrix $\lim_{t\to\infty} A_1^t = 0$.

For this to happen, we need that the absolute value of all the eigenvalues of A_1 are inside the unit circle.

Stability of the VAR(p)

Generalizing the stability property. We can rewrite a VAR(p) model using the lag polynomial as:

$$A(L)\mathbf{x_t} = \varepsilon_{\mathbf{t}}$$

where

$$A(L) = I - A_1 L - A_2 L^2 - \cdots - A_p L^p$$

and where for simplicity we have removed the constant.

Then, a VAR(p) process is said to be stable if all the roots of the equation:

$$det\,[A(L)] = |A(L)| = 0$$

are outside the unit circle (eigenvalues must be inside it).

Granger's non-Causality

Definition: $x_{1,t}$ is said to cause $x_{2,t}$ *in Granger's sense* if $x_{1,t}$ improves the forecast of $x_{2,t+j}$, for $j > 0$.

Example: Consider the bivariate VAR(1) model:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} a_1^0 \\ a_2^0 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

then $x_{1,t}$ is said *not to Granger-cause* $x_{2,t}$ if $a_{2,1} = 0$.
Note:

- Statistical tests are for non-causality (i.e. $H0 : a_{2,1} = 0$).
- Granger's non-causality is a property of the data, not of the model.
- Is a weak form of causality: entails only temporal dependence.

Cointegration

Assume to have two I(1) variables y_t and x_t which are known to be entirely unrelated and run the regression

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

You expect to estimate $\beta = 0$.

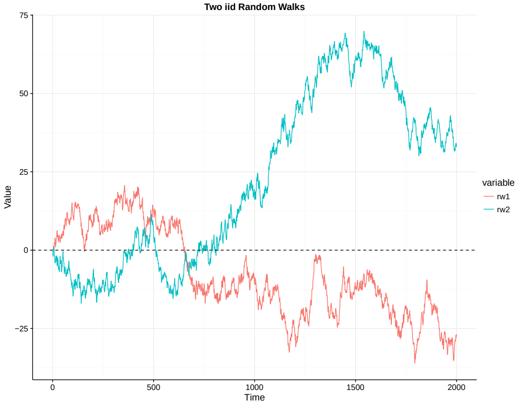
But...this is not what happens!

Why is it so?
Because the *exogeneity* assumption required by the Gauss-Markov condition does not hold! In general we need stationary variables for estimating with the OLS in an unbiased manner.

Cointegration

Example: generate y_t and x_t as two iid Random Walk so:

- we know they are I(1);
- we know they are unrelated.



- $\beta = -0.3$
- $R^2 = 0.48$
- $t_{stat} = -43$
- $p_{val} = 2^{-16}$

Cointegration

Therefore:
The exact question to ask when we have two I(1) variables is not whether they are correlated, but whether they are **cointegrated**.

Definition:
Assume the M variables of the vector $\mathbf{x_t}$ are all I(1). The variables are then said to be cointegrated if there exist one or more vectors α of dimensions $(M \times 1)$ such that $\alpha \mathbf{x_t} \sim I(0)$.

This means that if exists (at least) one vector α such that the linear combination of the I(1) variables is stationary; then the variables are said to be cointegrated.

Note

- cointegrated variables share a common trends;
- the vector α is called the cointegrating vector;

Testing for Cointegration

A test for cointegration is very intuitive.

Example: Take the bivariate VAR(1)

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} a_1^0 \\ a_2^0 \end{bmatrix} + \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

with $x_{1,t}, x_{2,t}$ that are both I(1) and cointegrated.
Then, if after having estimated the model we take the residuals $\varepsilon_{1,t}, \varepsilon_{2,t}$, they must be I(0).

We can test that by using the Dickey-Fuller test (see previous set of slides).

Conclusion

Why studying the VAR and the cointegration is important for macroeconomics?

- a multivariate analysis improves forecast
- a univariate model is too much simplistic and not descriptive
- VAR representation allows to analyse the Impulse Responses
- VAR representation allows to explain the variance of each variables by means of the others
- many macro variables have common trends and cointegration helps us detecting it

But...for this short tutorials it is sufficient for you to know the main concepts and to have an idea of why time series analysis is useful and how it can be applied to macroeconomics.

Further readings

Non mandatory...but if you want to work with time series they are STRONGLY ENCOURAGED!!!

Books (Intros and historical view)

- Time Series in general: Brockwell and Davis *Introduction to TS and Forecasting*
- Multivariate Time Series: Lutkepohl *Introduction to multiple TS Analysis*
- History of Econometrics: Louca *The years of high econometrics*

Seminal Papers

- Haavelmo (1944) *The probability approach in Econometrics*
- Koopmans (1950) *Statistical inference in dynamic economic models*
- Lucas (1976) *Econometric policy evaluation: a critique*
- Sims (1980) *Macroeconomics and reality*